May 3rd, 2017 **Name (Please Print)** ____________________

Special Topics in Probability - Final - Semester II

Instructions: Maximum time is 3 hours. Maximum Score is 100. If you are using a result shown in class then please state it precisely.

1. (**Nash Inequality**)

   (a) (3 points) Define what is meant to say "A weighted graph (Γ, μ) satisfies (N_α).

   (b) (10 points) Let Γ_i for $i = 1, 2$ be graphs, with natural weights, which satisfy (N_{α_i}) respectively. Show that the join of Γ_1 and Γ_2 satisfies $(N_{\alpha_1 \wedge \alpha_2})$

   (c) (12 points) Let Γ be two copies of $\mathbb{Z}^d$ joined at the origin. Let the transition kernel of the random walk on Γ be denoted by $p_n(x, y)$. Then show that

   $$p_n(x, y) \leq \frac{c_2}{n^{\frac{d}{2}}},$$

   for some $c_2 > 0$.

2. (**Isoperimetric Inequality**)

   (a) (3 points) Define what is meant to say "A weighted graph (Γ, μ) satisfies (I_α).

   (b) (10 points) Show that I_α is stable under rough isometry.

   (c) (12 points) Show that the Binary tree $\mathbb{T}^2$ satisfies (I_∞) with $C_0 = 3$.

3. (**Effective Resistance**) Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $B_0, B_1 \subset V$.

   (a) (3 points) Definte the effective conductance between B_0 and B_1.

   (b) (10 points) Let $B_0 = \{x\}$ and $B_1 = \{y\}$ where $x \sim y$ in V. Assume that the graph will be disconnected if we remove the edge $\{x, y\}$. Find $\mathrm{C_{eff}}(B_0, B_1)$.

   (c) (12 points) Consider the graph $\mathbb{Z}$ with natural weights. Calculate $\mathrm{R_{eff}}(\{0\}, [-n, n]^c)$. Use this to conclude that $\mathbb{Z}$ is recurrent.

4. (**Harmonic Functions**) Let $V = \mathbb{Z}^3$ be equipped with the cannonical edges and natural weights. Let X_n be the random walk on it. Let $n \geq 1$, $A = \mathbb{Z}^3 \setminus \{0\}$. Let $h_n, h : V \to [0, 1]$ be given by

   $$h_n(x) = \mathbb{P}^x(T_0 \geq n) = \mathbb{P}^x(X_k \in A, 1 \leq k \leq n)$$

   and

   $$h(x) = \mathbb{P}^x(T_0 = \infty) = \mathbb{P}^x(X_n \in A, \text{ for all } n \geq 0).$$

   (a) (5 points) State the maximum principle for Harmonic functions.

   (b) (5 points) Show that $h_n = Q^n 1_A$ and $h = Qh$, where Q is the restriction of P onto A.

   (c) (5 points) Suppose $\alpha = \sup_{x \in A} h(x)$, show that $0 < \alpha \leq 1$ and $h \leq \alpha 1_A$

   (d) (5 points) Using (a) and (b), conclude that $h \leq \alpha h_n$

   (e) (5 points) Conclude that $\max_{x \in \partial A} h(x) \neq \sup_{x \in \bar{A}} h(x)$.